Perfectoid Spaces

10.10.2024
Linus Rösler

$\boxed{\text{Last time}}$: $\mathrm{Spa}(\underbrace{\mathbb{Q}_p^{nr}\langle x\rangle}_{=K\langle x\rangle}, \underbrace{\mathcal{O}_{\mathbb{Q}_p^{nr}}\langle x\rangle}_{=A\langle x\rangle}) \xrightarrow{\mathrm{Supp}} \mathrm{Spec}\, K\langle x\rangle$, $\quad |\cdot|_x \longmapsto \mathrm{Supp}\,|\cdot|_x$

$\mathrm{Spa}(\mathbb{Q}_p\langle x\rangle, \mathbb{Z}_p\langle x\rangle)$

We saw $\mathrm{Supp}^{-1}(\mathrm{Spec}\, K\langle x\rangle \setminus \{0\}) \overset{\text{top.}}{\cong} \overset{\text{punctured}}{\overline{B(0,1)}} \cong A$

<u>Forgot</u>: for $R\left(\frac{f_1,\dots,f_n}{g}\right)$ we have to require $(f_1,\dots,f_n) \underset{\text{open}}{\subseteq}$ "ring"

$\hookrightarrow$ $R\left(\frac{x}{p}\right)$ from last time is not allowed! But one can simply change it to $R\left(\frac{p,x}{p}\right)$.

$\boxed{\text{Now}}$: $\mathrm{Supp}^{-1}(\{(0)\}) = ?$

Take $|\cdot|$ with $\mathrm{Supp}\,|\cdot| = (0)$. By def. of Spa, $|A\langle x\rangle| \le 1 \to R_{|\cdot|} \supseteq A\langle x\rangle$

We have the prime ideal $\mathfrak{p}_{|\cdot|} \subseteq R_{|\cdot|}$, and then also $\mathfrak{q} = \mathfrak{p}_{|\cdot|} \cap A\langle x\rangle \underset{\text{prime}}{\subseteq} A\langle x\rangle$.

<u>Note</u>: Krull dim $A\langle x\rangle = 2$. Consider $\mathrm{Spec}\, A\langle x\rangle \longrightarrow \mathrm{Spec}\, A = \{\eta, (p)\}$ (A DVR)

If $\mathfrak{q}$ maps to η, then $\mathfrak{q} \in \mathrm{Spec}\, K\langle x\rangle$. <u>But</u> $\mathfrak{q}$ open $\Rightarrow$ ↯

$\hookrightarrow$ $\mathfrak{q}$ maps to (p) $\Rightarrow \mathfrak{q} = (p, \dots) \Rightarrow \bar{\mathfrak{q}} \in \mathrm{Spec}\left(A\langle x\rangle/p\right) \cong \mathrm{Spec}(\mathbb{F}_p[x])$

$\Rightarrow \bar{\mathfrak{q}} = (0)$ or $(x-\lambda)$ $\Rightarrow \mathfrak{q} = (p)$ or $(p, x-\lambda)$.

<u>Case 1</u>: $\mathfrak{q} = (p)$. For $a \in A\langle x\rangle \setminus (p) \Rightarrow |a| = 1$

$\hookrightarrow$ <u>so</u>: $a \in A\langle x\rangle \Rightarrow a = p^n b$, $|b| = 1$. So if such a $|\cdot|$ exists, it must be $|\cdot|_p$, and indeed this exists.

<u>Case 2</u>: $\mathfrak{q} = (p, x-\lambda)$. By a similar argument, for $a \in A\langle x\rangle_{\mathfrak{q}}$: $a = p^n (x-\lambda)^m b$ with $|b| = 1$. $\Rightarrow |a| = |p|^n \cdot |x-\lambda|^m$

<u>Recall</u>: $\underset{\text{"}|\cdot|\text{"}}{|\cdot|_x} : K\langle x\rangle \longrightarrow \Gamma_x$ ← totally ordered abelian group

$\boxed{\text{Def}}$ (rank): Let Γ be a totally ordered abelian group. Then the rank of Γ is defined as

$\mathrm{rk}\,\Gamma := \sup\{n \ge 0 \mid \exists\, \Gamma_0, \dots, \Gamma_n \le \Gamma$ convex subgrps of Γ with $0 = \Gamma_0 \subsetneq \Gamma_1 \subsetneq \dots \subsetneq \Gamma_n = \Gamma\}$

A subgroup $\Sigma \le \Gamma$ is convex iff $\forall a,b \in \Sigma, c \in \Gamma$: $a < c < b \Rightarrow c \in \Sigma$.

$\boxed{\text{HW}}$: $\mathrm{rk}\,\Gamma_x = 1$ or 2

$\boxed{\text{Prop}}$: Let Γ be a totally ordered ab. grp. TFAE: (additive notation)

[(1) $\mathrm{rk}\,\Gamma = 1$] $\qquad$ [(3) $\Gamma \hookrightarrow \mathbb{R}$.]

[(2) $\forall a,b,\ a>0, b>0$: $\exists n \in \mathbb{Z}_{>0}$: $b < na$]

Idea of the pf ① (or ②) $\Rightarrow$ ③: Choose $a \in \Gamma \setminus \{0\}$ and send $\varphi(a) = 1$ ($\varphi : \Gamma \to \mathbb{R}$).

$\Rightarrow \varphi(na) = n$. If $0 < b \in \Gamma$, then $\forall\, m \in \mathbb{Z}_{>0}$: $n_m := \min\{n > 0 \mid mb < na\}$

Show: $\lim_{m\to\infty} \frac{n_m}{m}$ exists, $\varphi(b) =$ this limit.

$\boxed{\text{HW9}}$: Finish the proof, i.e. φ is a homom. of ordered abelian groups.

So if $\operatorname{rk} \Gamma_x = 1$: $\Rightarrow$ WMA $\Gamma_x = \mathbb{R}$, & WMA $|p| = \frac{1}{p}$

$\Rightarrow |x - \lambda| = \gamma \in \mathbb{R} \Rightarrow 0 < \gamma < 1$.

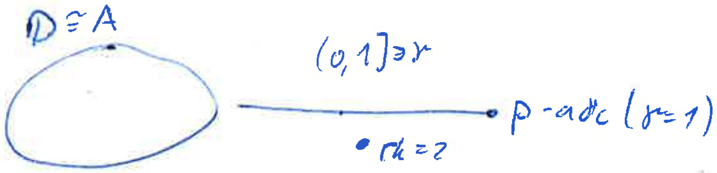
So the valuation (if it exists) must be: for $f \in K\langle x\rangle$, write $f = \sum_{i\in\mathbb{N}} a_i (x-\lambda)^i$

$\hookrightarrow |f|_x = \max_{i\in\mathbb{N}} |a_i|_p \cdot \gamma^i$, and this works! (Gauss valuation)

If $\operatorname{rk} \Gamma_x = 2$: $\Rightarrow$ WMA $\Gamma_x = \mathbb{R} \times \gamma^{\mathbb{Z}}$ (lexicographic), $|p| = (\frac{1}{p}, 0)$, $|x - \lambda| = (0, \gamma)$

$\hookrightarrow \left|\sum_{i\in\mathbb{N}} a_i (x-\lambda)^i\right| = \max_{i\in\mathbb{N}} (|a_i|_p, \gamma^i)$.

Spa$(K\langle x\rangle, A\langle x\rangle)$:



$\boxed{\text{Ex}}$: formal affine line $/A$: $\operatorname{Spa}(A\langle x\rangle, A\langle x\rangle) \underset{\text{open?}}{\supseteq} \operatorname{Spa}(K\langle x\rangle, A\langle x\rangle)$

But more val's: w/ supp (p) or $(p, x-\lambda)$ $\longleftrightarrow$ valuation on $\operatorname{Frac}(A\langle x\rangle / \text{supp})$

$\longleftrightarrow$ val's on $\overline{\mathbb{F}}_p(x)$ or $\overline{\mathbb{F}}_p$

$\hookrightarrow$ if Supp $= (p, x-\lambda)$: $|\cdot|_{1,\overline{\mathbb{F}}_p} \circ \left(A\langle x\rangle \xrightarrow[\text{ev}_\lambda \,(\text{mod } p)]{} \overline{\mathbb{F}}_p\right)$

classical closed pts of the formal scheme.

$\hookrightarrow$ if Supp $= (p)$: ① $|\cdot|_{1,\overline{\mathbb{F}}_p(x)} \circ (\text{mod } p)$

② $|\cdot|_{x-\lambda, \overline{\mathbb{F}}_p(x)} \circ (\text{mod } p)$

$\boxed{\text{Next goal}}$: structure pre-sheaf

$\boxed{\text{Ex}}$: $R\left(\frac{p, x}{p}\right) \underset{\text{open}}{\subseteq} \operatorname{Spa}(\overset{\cong \mathbb{Q}_p}{K\langle x\rangle}, A\langle x\rangle) = X$

$\cup$ classical pts

$p \cdot A \underset{\text{open}}{\subseteq} \mathbb{D} \cong A$ $\longrightarrow$ What should be $\mathcal{O}_X\left(R\left(\frac{p,x}{p}\right)\right)$?

$\{\lambda \in \mathbb{D} \mid |\lambda|_p = |p|_p\}$ (and $\mathcal{O}_X^+\left(R\left(\frac{p,x}{x}\right)\right)$)

Perfectoid spaces

10.10.2024
Linus Rösler

If life, universe and everything (including God) had a plan, then

$$\mathcal{O}_X\left(R\left(\frac{p,x}{x}\right)\right) = K\langle x\rangle\Big(\overbrace{\underbrace{\tfrac{p,x}{p}}_{\text{what we localize}}}^{\text{topology}}\Big) \quad \text{(or rather completed version } \langle\ \rangle\text{)}.$$

↳ But what is this?

Localization for non-archimedean rings:

A non-arch., fix also $T_i \underset{\text{subsets}}{\subseteq} A$ $(i \in I)$ (in our ex. $I = \{1\}$, $T_1 = \{p,x\}$)

Condition (*): $\forall\, 0 \in U \underset{\text{open}}{\subseteq} A$, $\forall i \in I$ $\forall n \in \mathbb{N}$:

$$T_i^n \cdot U = \underbrace{\Big\langle \Big(\prod_{j=1}^{n} a_j\Big)\cdot u \;\Big|\; a_j \in T_i,\ u \in U \Big\rangle_+}_{\text{generated additive subgroup}} \underset{\text{open}}{\subseteq} A$$

Ex: $\mathbb{Q}_p\langle x\rangle = A$, $I = \{1\}$, $T_1 = \{x\}$, $U = \mathbb{Z}_p\langle x\rangle$, $T_1 \cdot U = (x)_{\mathbb{Z}_p\langle x\rangle}$ not open!!

Ex: if $\{x\} \rightsquigarrow \{p,x\}$ then (*) satisfied

$\boxed{\text{Prop}}$ (localization exists): A, T_i as above (with (*) sat'd). Fix also $s_i \in A$ for every $i \in I$, $R =$ mult. subset of A gen'd by the s_i's.

Then: $\exists!$ non-arch ring structure $A\left(\frac{T_i}{s_i} \,\middle|\, i \in I\right)$ on $R^{-1}A$ satisfying

(i) $\varphi : A \xrightarrow[\text{map}]{\text{usual}} A\left(\frac{T_i}{s_i} \,\middle|\, i \in I\right)$ continuous and $\left\{ \frac{\varphi(t)}{\varphi(s_i)} \,\middle|\, i \in I,\ t \in T_i \right\}$ is power-bounded

(ii) Universal property (next time)